SOME FURTHER APPLICATIONS OF FINITE DIFFERENCE OPERATORS

TECHNICAL REPORT NO. 3

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KAI-TAI FANG
STANFORD UNIVERSITY

and

INSTITUTE OF APPLIED MATHEMATICS
ACADEMIA SINICA, BEIJING, CHINA

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Some Further Applications of Finite Difference Operators

Kai-Tai Fang*
Stanford University

and

Institute of Applied Mathematics
Academia Sinica, Beijing, China

1. INTRODUCTION

Recently R. F. Link (1981) used finite difference operators for a unified approach to determining the moments of discrete probability distributions.

He presented derivations for several well-known distributions: the binomial, Poisson, geometric, and hypergeometric. Following his approach we give derivations for several important distributions in occupancy problems and some other well-known discrete distributions (negative binomial, power series, and factorial series). Also, Johnson and Kotz (1981) gave the formulae of (descending or ascending) factorial moments for the binomial, Poisson, hypergeometric, negative binomial, and negative hypergeometric.

If the moments about the origin are expressed by means of finite difference operators, the corresponding descending factorial moments can be found immediately by a theorem given below.

In this article, E is the displacement operator and Δ is the difference operator; i.e.

$$Ef(x) = f(x+1) \text{ and } \Delta f(x) = f(x+1) - f(x).$$

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The relationship between E and Δ is:

$$\Delta \equiv (E-1) \ ; \quad E \equiv (1+\Delta).$$

If X is a discrete random variable and

$$p_j = P(X=j) \quad j = 0, 1, 2, \ldots$$

Its s^{th} moment about the origin, μ'_s, is

$$\mu'_s = \Sigma_{j=0}^{\infty} \ p_j \ j^s$$

and its sth descending factorial moments $\mu^{(s)}$ is

$$\mu^{(s)} = \Sigma_{j=0}^{\infty} \ p_j \ j^{(s)} = \Sigma_{j=s}^{\infty} \ p_j \ j^{(s)}$$

where $j^{(s)} = j\ (j-1) \ \ldots \ (j-s+1)$.

In what follows, $\binom{a}{b} = 0$, for $a<b$ or $b<0$, and $\Delta^n \underset{\sim}{0}^m = \Delta^n x^m|_{x=0}$.

2. THE CLASSICAL OCCUPANCY PROBLEMS

The basic models of the classical occupancy problems are as follows: there are m urns and n balls, and the balls are distributed among the m urns. Let M_t be the number of urns containing exactly t balls ($t=0, 1, \ldots, n$). Find the distributions of M_t (cf. Johnson and Kotz, 1977).

According to whether the urns and balls are distinguishable (DT) or indistinguishable (IDT), and whether empty urns are permitted or not, there are several possible models, four of which are listed in Table 1, where $T_{\{i\}}(n,m)$ denotes the number of ways in which n balls are distributed among m urns under the model {i} (cf. Johnson and Kotz, 1977, p.37).

Table 1. The Classical Occupancy Models

Model	Urns	Balls	Are empty urns permitted	T{i} (n, m)
{1}	DT	IDT	Yes	$\binom{n+m-1}{m-1}$
{2}	DT	IDT	No	$\binom{n-1}{m-1}$
{3}	DT	DT	Yes	m^n
{4}	DT	DT	No	$\Delta^m \, \underset{\sim}{0}^n$

The models {1} and {2} correspond to the Bose - Einstein system and the models {3} and {4} correspond to Maxwell - Boltzmann system. Let $M_t^{\{i\}}$ be M_t under the model {i} (i=1, 2, 3, 4). It is well-known that the distributions of $M_t^{\{i\}}$ are:

$$P\{M_t^{\{1\}} = r\} = \frac{\binom{m}{r}}{\binom{n+m-1}{m-1}} \sum_{j=0}^{m-r} (-1)^j \binom{m-r}{j} \binom{n+m-(r+j)(t+1)-1}{m-r-j-1}$$

$$P\{M_t^{\{2\}} = r\} = \frac{\binom{m}{r}}{\binom{n-1}{m-1}} \sum_{j=0}^{m-r} (-1)^j \binom{m-r}{j} \binom{n-(r+j)t-1}{m-r-j-1}$$

$$P\{M_t^{\{3\}} = r\} = \frac{\binom{m}{r}}{m^n} \sum_{j=0}^{m-r} (-1)^j \binom{m-r}{j} \frac{n!}{(t!)^{r+j}(n-(r+j)t)!} (m-r-j)^{n-(r+j)t}$$

$$P\{M_t^{\{4\}} = r\} = \frac{\binom{m}{r}}{\Delta^m 0^n} \sum_{j=0}^{m-r} (-1)^j \binom{m-r}{j} \frac{n!}{(t!)^{r+j}(n-(r+j)t)!} \Delta^{m-r-j} \underset{\sim}{0}^{n-(r+j)t}$$

We can express these in a unified form:

$$P\{M_t^{\{i\}} = r\} = \frac{\binom{m}{r}}{T_{\{i\}}(n,m)} \sum_{j=0}^{m-t} (-1)^j \quad T_{\{i\}} (n-(r+j)t, m-r-j)\ h_i\ (n,t,r+j) \qquad \text{for } i=1,2,3,4,$$

where

$$h_1(n,t,\alpha) = h_2(n,t,\alpha) = 1$$
$$h_3(n,t,\alpha) = h_4(n,t,\alpha) = \frac{n!}{(t!)^{\alpha}(n-\alpha t)!} \quad .$$

Let $\mu'_{\{i\}s}$ be the sth moment of $M_t^{\{i\}}$ about the orgin. We can derive $\mu'_{\{i\}s}$ by using finite difference operators. From the definition of moments, we have

$$\mu'_{\{i\}s} = \sum_{r=0}^{m} r^s \ P\{M_t^{\{i\}} = r\} = \sum_{r=1}^{m} P\{M_t^{\{i\}} = r\} E^r \underset{\sim}{0}^s$$

and we can express $P\{M_t^{\{i\}} = r\}$ as

$$T_{\{i\}} (n,m)\ P\{M_t^{\{i\}} = r\} = \sum_{j=r}^{m} (-1)^{r+j} \binom{j}{r}\binom{m}{j} T_{\{i\}} (n-jt, m-j)\ h_i(n,t,j)\ .$$

Thus we get

$$T_{\{i\}} (n,m)\ \mu'_{\{i\}s} = \sum_{r=1}^{m} \sum_{j=r}^{m} (-1)^{r+j} \binom{j}{r}\binom{m}{j} T_{\{i\}} (n-jt, m-j)\ h_i(n,t,j)\ E^r \underset{\sim}{0}^s$$

$$= \sum_{j=1}^{m} (-1)^j \binom{m}{j} T_{\{i\}} (n-jt,\ m-j)\ h_i\ (n,t,j) \sum_{r=1}^{j} (-1)^r \binom{j}{r} E^r \underset{\sim}{0}^s$$

$$= \sum_{j=1}^{m} (-1)^j \binom{m}{j} T_{\{i\}} (n-jt,\ m-j)\ h_i\ (n,t,j)\ (1-E)^j \underset{\sim}{0}^s$$

$$= \sum_{j=1}^{s} \binom{m}{j} T_{\{i\}} (n-jt,\ m-j)\ h_i\ (n,t,j)\ \Delta^j \underset{\sim}{0}^s, \qquad \text{for } i=1.\ 2,\ 3,\ 4,$$

i.e.,

$$(2.1)\quad \mu'_{\{i\}s} = \sum_{j=1}^{s} \binom{m}{j} T_{\{i\}}(n-jt,\ m-j)\ h_i(n,t,j)\ \Delta^j\ \underset{\sim}{0}^s / T_{\{i\}}(n,m).$$

3. THE RESTRICTED OCCUPANCY PROBLEMS

In the models above, if the maximum number of balls allowed in any urn is k, we get the restricted occupancy problems. The models are as follows:

There are m urns and n balls. Each urn contains k cells. The n balls are assigned among the m urns so that each cell contains at most one ball.

Fang (1982) considers several models according to whether urns, balls and cells are DT or IDT, and whether empty urns are permitted or not. Ten such models are listed in Table 2.

Fang (1982) obtained the distributions of M_t under the model (i), i=1, 2, ..., 10. They have the following unified form except for model (9):

$$P\{M_t=r\} = \frac{\binom{m}{r}}{T_i(n,m,k)} \sum_{j=0}^{m-r} (-1)^j \binom{m-r}{j} T_i\,(n-(r+j)t,\ \ m-(r+j),\ k)\ g_i(n,k,t,r+j)$$

where $T_i(n,m,k)$ is the number of ways of distributing n balls among m urns under the model (i) and

$$g_i(n,k,t,\alpha) = \begin{cases} 1 & \text{for } i = 1,\ 2, \\ \binom{k}{t}^{\alpha} & \text{for } i = 3,\ 4, \\ \binom{k}{t}^{\alpha} n^{(\alpha t)} & \text{for } i = 5,\ 6, \\ \dfrac{n!}{(t!)^{\alpha}(n-\alpha t)!} & \text{for } i = 7,\ 8, \\ \dfrac{n!}{m^{(\alpha)}(t!)^{\alpha}(n-\alpha t)!} & \text{for } i = 10. \end{cases}$$

Table 2. Models of the Restricted Occupancy Problems

Symbol if empty urns are permitted	Symbol if empty urns are not permitted	urns	balls	cells
(1)	(2)	DT	IDT	IDT
(3)	(4)	DT	IDT	DT
(5)	(6)	DT	DT	DT
(7)	(8)	DT	DT	IDT
(9)	(10)	IDT	DT	IDT

We see that the distributions of M_t have a form which is similar to that of the classical models. Thus

$$(3.1)\quad \mu'_{(i)_s} = \sum_{j=1}^{s} \binom{m}{j} T_i\ (n-jt,\ m-j,k)\ g_i(n,k,t,j)\Delta^{\alpha}0^s/T_i(n,m,k) \quad i=1,2,\ldots,8.10,$$

where $\mu'_{(i)_s} = E\ (M_t^s)$ under the model (i).

4. COMMITTEE PROBLEMS

A group contains n individuals, any w_i of whom can be selected at random to form the i^{th} committee (i=1, 2, ..., r). Find the probability that exactly m individuals will be committee members. This has been called the committee problem which has been studies by several authors (cf. Johnson and Kotz, (1977), and Holst (1980)).

Let L be the number of individuals who do not serve on any of the r committees; then exactly (n-L) individuals will be committee members.

We can find the distribution of L by using the finite difference operator (White, 1971) or by the inclusion-exclusion principle (Sprott, 1969). It is

$$P\{L=k\} = \sum_{j=0}^{n-k} (-1)^j \binom{k+j}{k}\binom{n}{k+j} \prod_{i=1}^{r} \binom{m-k-j}{w_i} \Big/ \binom{n}{w_i}.$$

Its s^{th} moment μ_s' is

$$(4.1) \qquad \mu_s' = \sum_{k=1}^{n-\max w_i} \sum_{j=0}^{n-k} (-1)^j \binom{k+j}{k}\binom{n}{k+j}\left(\prod_{i=1}^{r} \binom{n-k-j}{w_i} \Big/ \binom{n}{w_i}\right) E^k \underset{\sim}{0}^s$$

$$= \sum_{k=1}^{n} \sum_{\alpha=k}^{n} (-1)^{\alpha+k} \binom{\alpha}{k}\binom{n}{\alpha}\left(\prod_{i=1}^{r} \binom{n-\alpha}{w_i} \Big/ \binom{n}{w_i}\right) E^k \underset{\sim}{0}^s$$

$$= \sum_{\alpha=1}^{n} (-1)^\alpha \binom{n}{\alpha}\left(\prod_{i=1}^{r} \binom{n-\alpha}{w_i} \Big/ \binom{n}{w_i}\right) \sum_{k=1}^{\alpha} (-1)^k \binom{\alpha}{k} E^k \underset{\sim}{0}^s$$

$$= \sum_{\alpha=1}^{s} \binom{n}{\alpha}\left(\prod_{i=1}^{r} \binom{n-\alpha}{w_i} \Big/ \binom{n}{w_i}\right) \Delta^\alpha \underset{\sim}{0}^s.$$

In particular, if $w_i = w_2 = \ldots = w_r = w$, we have

$$(4.2) \qquad \mu_s' = \sum_{\alpha=1}^{s} \binom{n}{\alpha}\binom{n-\alpha}{w}^r \Delta^\alpha \underset{\sim}{0}^s \Big/ \binom{n}{w}^r.$$

5. OTHER DISTRIBUTIONS

(1) Negative Binomial Distribution

The formula for the moments of the negative binomial distribution as obtained by Link (1981) contains double summations, and Link suggests that a way should be found to get rid of these .

We can express the negative binomial distribution as

$$p_j = \binom{k+j-1}{k-1} q^j p^k \qquad j=0,\ 1,\ 2,\ \ldots$$

(Patel, Kapadia and Owen, 1976, p.22). Now

$$(5.1) \qquad \mu_s' = \Sigma_{j=0}^{\infty} \binom{k+j-1}{k-1} q^j p^k E^j \underset{\sim}{0}^s$$

$$= p^k (1-qE)^{-k} \underset{\sim}{0}^s$$

$$= p^k (p-q\Delta)^{-k} \underset{\sim}{0}^s$$

$$= (1 - q/p\Delta)^{-k} \underset{\sim}{0}^s$$

$$= \sum_{j=0}^{s} \binom{k+j-1}{k-1} (q/p)^j \Delta^j \underset{\sim}{0}^s .$$

(2) Power Series Distribution

Here $p_j = a_j \theta^j / f(\theta) \qquad j=0,\ 1,\ 2,\ \ldots$

Where $f(\theta) = \Sigma_{j=0}^{\infty} a_j \theta^j$.

The class of power series distributions includes the Poisson, binomial, negative binomial distributions. From the definition, we have

$$(5.2) \qquad \mu_s' = \sum_{j=1}^{\infty} a_j \theta^j E^j \underset{\sim}{0}^s / f(\theta)$$

$$= \sum_{j=1}^{\infty} a_j \theta^j \sum_{i=0}^{j} \binom{j}{i} \Delta^i \underset{\sim}{0}^s / f(\theta).$$

$$= \sum_{i=1}^{s} \sum_{j=i}^{\infty} a_j \theta^j \binom{j}{i} \Delta^i \underset{\sim}{0}^s / f(\theta)$$

$$= \sum_{i=1}^{s} \frac{\theta^i}{i!} f^{(i)} \Delta^i \underset{\sim}{0}^s / f(\theta)$$

where $f^{(i)}(\theta)$ denotes the ith derivative.

(3) Factorial Series Distributions

$$p_j = \frac{\theta^{(j)}}{j!} \frac{\Delta^j f(\underset{\sim}{0})}{f(\theta)} \qquad j = 0, 1, 2, \ldots$$

where $f(\theta) = \sum_{j=0}^{\infty} \frac{\theta^{(j)}}{j!} \Delta^j f(\underset{\sim}{0})$ and $f(\theta)$ satisfies suitable conditions (cf. Johnson and Kotz, 1977, pp.87-88). We have

$$(5.3) \qquad f(\theta)\, \mu'_s = \sum_{j=0}^{\infty} \frac{\theta^{(j)}}{j!} \Delta^j f(\underset{\sim}{0})\, E^j \underset{\sim}{0}^s$$

$$= \sum_{j=0}^{\infty} \frac{\theta^{(j)}}{j!} \Delta^j f(\underset{\sim}{0}) \sum_{i=0}^{j} \binom{j}{i} \Delta^i \underset{\sim}{0}^s$$

$$= \sum_{i=1}^{s} \sum_{j=i}^{\infty} \frac{\theta^{(j)}}{(j-i)!} \Delta^j f(\underset{\sim}{0}) \frac{\Delta^i \underset{\sim}{0}^s}{i!}$$

$$= \sum_{i=1}^{s} \sum_{\alpha=0}^{\infty} \frac{\theta^{(\alpha+i)}}{\alpha!} \Delta^{\alpha+i} f(\underset{\sim}{0}) \frac{\Delta^i \underset{\sim}{0}^s}{i!}$$

$$= \sum_{i=1}^{s} \theta^{(i)} \Delta^i \sum_{\alpha=0}^{\infty} \frac{(\theta-i)^{(\alpha)}}{\alpha!} \Delta^\alpha f(\underset{\sim}{0}) \frac{\Delta^i \underset{\sim}{0}^s}{i!}$$

$$= \sum_{i=1}^{s} \theta^{(i)} \Delta^i f(\theta-i) \frac{\Delta^i \underset{\sim}{0}^s}{i!} .$$

6. DESCENDING FACTORIAL MOMENTS

The formulas which have been found by Link, Johnson and Kotz, and by us can all be written in the following form:

$$\mu'_s = \sum_{i=1}^{s} a_i \Delta^i \underset{\sim}{0}^s. \qquad s = 1, 2, \ldots$$

It is well known (cf. Johnson and Kotz, 1977, pp. 8-10) that:

$$(6.1) \qquad X^s = \sum_{j=1}^{s} X^{(j)} \Delta^j \underset{\sim}{0}^s / j!,$$

thus there is a relationship between μ'_s and $\mu^{(s)}$ which is

$$(6.2) \qquad \mu'_s = \sum_{j=1}^{s} \mu^{(j)} \Delta^j \underset{\sim}{0}^s / j!$$

Let $A = (a_{ij})$ be a s x s matrix, where

$$a_{ij} = \begin{cases} \Delta^j \underset{\sim}{0}^i / j! & \text{for } i \geq j \\ 0 & \text{for } i < j. \end{cases}$$

Obviously, A is a lower triangular matrix with $a_{ii} = 1$ ($i=1,2,\ldots,s$). Hence A is non-degenerate. From this fact, we obtain the following theorem.

Theorem If the moments about the origin have the form

$$\mu'_s = \sum_{j=1}^{s} a_j \Delta^j \underset{\sim}{0}^s \qquad s = 1, 2, \ldots$$

then the corresponding descending factorial moments are

$$\mu^{(s)} = a_s s! \qquad s = 1, 2, \ldots$$

By using the theorem and formulas obtained by Link, Johnson and Kotz, and us, we immediately derive the descending factorial moments of the following distributions. These moments can be found in textbooks or papers, but the derivations are often elaborate or lengthy.

(1) Binomial Distribution

$$\mu^{(s)} = n^{(s)} p^s.$$

(2) Poisson Distribution

$$\mu^{(s)} = t^s.$$

(3) Geometric Distrbution

$$\mu^{(s)} = s! \, q^{s-1} / p^s.$$

(4) Hypergeometric Distribution

$$\mu^{(s)} = n^{(s)} X^{(s)} / N^{(s)}.$$

(5) Negative Binomial Distribution

$$\mu^{(s)} = (k+s-1)^{(s)} (q/p)^{s} = k^{[s]} (q/p)^{s},$$

where $k^{[s]} = k (k+1) \ldots (k+s-1)$.

(6) Power Series Distributions

$$\mu^{(s)} = \theta^{s} f^{(s)} (\theta) / f(\theta).$$

(7) Factorial Series Distributions

$$\mu^{(s)} = \theta^{(s)} \Delta^{s} f (\theta-s) / f(\theta).$$

(8) The Classical Occupancy Problems

$$\mu^{(s)}_{\{i\}} = m^{(s)} T_{\{i\}} (n-st, m-s) h_i(n,t,s) / T_{\{i\}} (n,m), \quad i=1,2,3,4.$$

(9) The Restricted Occupancy Problems

$$\mu^{(s)}_{(i)} = m^{(s)} T_i (n-st, m-s, k) g_i(n,k,t,s) / T_i (n,m,k), \quad i=1,2,\ldots,8,10.$$

Fang (1981) obtained the same results by using characteristic functions.

(10) Committee Problems

$$\mu^{(s)} = n^{(s)} \prod_{i=1}^{r} \binom{n-s}{w_i} / \binom{n}{w_i}.$$

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R. F. Link described a standard method of deriving the moments of the better-known discrete probability distribution functions by using finite difference operators. In this report these formulas are used to derive the moments of some basic distributions in occupancy problems and other known discrete distributions.